Cross-Over Options

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Preview

- A novel class of path-dependent options – cross-over (CO) options
- A CO option is a type of volatility instrument
- A CO option can be robustly replicated under very general conditions
  - Model-free – general price process, including jumps
  - Exact at any frequency – no discretization or jump errors.
- A vanilla option is a special case of CO options. This connection produces a number of new results and applications for vanilla options
  - A new model-free replication strategy for vanilla options
  - A new fundamental decomposition of vanilla option value into two parts: (1) due to continuous moves and (2) due to jumps
- Many potential applications

Setup

- A frictionless, arbitrage-free market with a single risky asset over $[0, T]$
- F_t is the forward price of the risky asset. Can ignore dividends, risk-free rate
- $\mathcal{T} = \{t_0, t_1, \dots, t_n\}$ is a monitoring partition, where $0 = t_0 < t_1 < \dots < t_n = T$
- $\Delta t := \max_i (t_i - t_{i-1})$
- $M_t(K,T) := \begin{cases} P_t(K,T) & \text{if} \quad K \le F_0 \\ C_t(K,T) & \text{if} \quad K > F_0 \end{cases}$

  where $P_t(K,T)$ and $C_t(K,T)$ are prices of European put and call with strike K and maturity T

- A CO option has a barrier K and expires at time-T with payoff

$$\Phi_T = \Phi_T(K, \mathcal{T}) := \sum_{i=1}^{n} B(F_{i-1}, F_i, K)\,|F_i - K|, \qquad \text{where}$$

$$B(F_{i-1}, F_i, K) := \begin{cases} 1 & \text{if } U(F_{i-1} - K) + U(F_i - K) = 1 \\ 0 & \text{otherwise} \end{cases}$$

indicates whether barrier is crossed over $[t_{i-1}, t_i]$ and $U(x) := 1_{\{x>0\}}$ (for "Up")

- Every time barrier K is crossed over (from above or below), payoff function gets increased by amount of "overshoot" $|F_i - K|$:

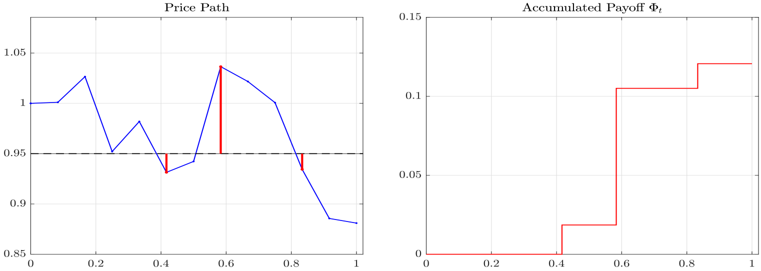


FIGURE: CO payoff with $T = 1$ year, $K = 0.95$, Δt = 1-month. Φ_T = Sum of lengths of red stems.

CO Option at Different Frequencies

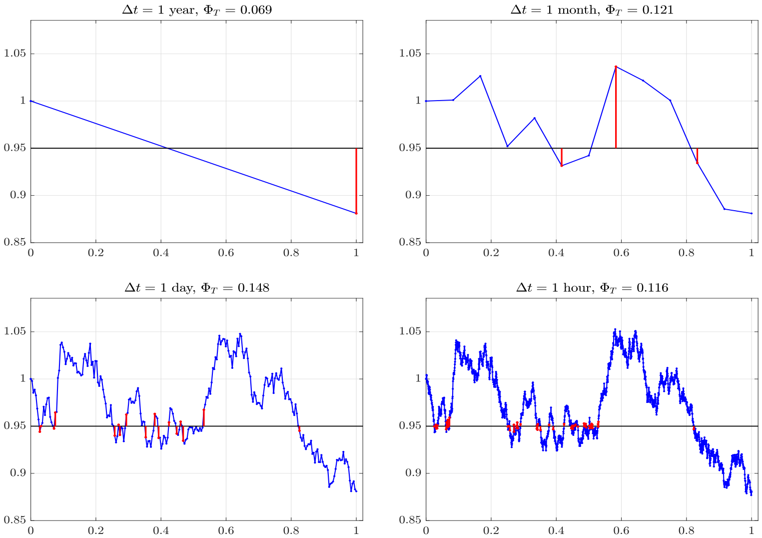


FIGURE: CO payoff with $T = 1$ year and $K = 0.95$. Monitoring frequency $\Delta t = 1$ year, 1 month, 1 day, and 1 hour. $\Phi_T =$ Sum of lengths of red stems.

CO Option at Different Frequencies

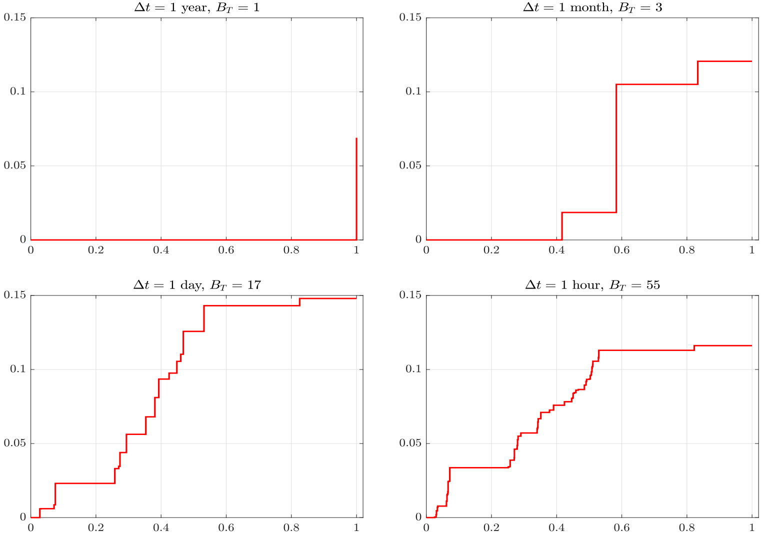


FIGURE: Accumulated payoff Φ_t of the CO option with $T = 1$ year and barrier $K = 0.95$. Monitoring frequency $\Delta t = 1$ year, 1 month, 1 day, and 1 hour. The realized number of crossing is $B_T = B_T(K, \mathcal{T}) := \sum_{i=1}^{n} Crossed_i$.

- Realized payoff Φ_T depends on specific price path and partition $\mathcal{T}$
- For smaller Δt, crossings are more frequent, but overshoots are smaller
- Remarkably, the market price of CO option, $CO_0(K) = E_0^{\mathbb{Q}}[\Phi_T]$, does **not** depend on $\mathcal{T}$
- This is because CO payoff satisfies certain Aggregation Property (AP)

Aggregation Property

Bondarenko (2014, JE):

$H(x,y)$ satisfies Aggregation Property (AP), if for any martingale X_t and for any times $0 \leq t \leq s \leq u \leq T$,

AP: $E_t^{\mathbb{Q}}[H(X_t, X_u)] = E_t^{\mathbb{Q}}[H(X_t, X_s)] + E_t^{\mathbb{Q}}[H(X_s, X_u)].$

If $H(x,y)$ satisfies AP, then discretely-sampled payoff $\sum_{i=1}^{n} H(F_{i-1}, F_i)$ has same market price as time-T payoff $H(F_0, F_T)$:

$$E_0^{\mathbb{Q}}\left[\sum_{i=1}^{n} H(F_{i-1}, F_i)\right] = E_0^{\mathbb{Q}}\left[H(F_0, F_T)\right].$$

- Payoffs that satisfy AP are rare, but special
- Important for variance trading, which relies on two key insights:
  1) Reduce path-*dependent* payoff to path-*independent* one – need AP
  2) Replicate path-*independent* payoff with a static portfolio of vanilla options – need Carr and Madan (1998) spanning formula
- Payoffs that satisfy AP can be *robustly* replicated
- A replication strategy is *robust*, if it
  1) is model-free, including jumps
  2) holds for *any* partition $\mathcal{T}$ (non-equidistant, non-small Δt)
  3) consists of two parts:
     (I) a *static* position in a portfolio of vanilla options
     (II) a *discrete* dynamic trading in underlying on dates in $\mathcal{T}$

- Want to price a contract which pays *discretely-sampled* realized variance:

$$RV_T^{(1)} = RV^{(1)}(\mathcal{T}) := \sum_{i=1}^{n} r_i^2,$$
$$RV_T^{(2)} = RV^{(2)}(\mathcal{T}) := \sum_{i=1}^{n} x_i^2,$$

where $r_i = \log\left(\frac{F_i}{F_{i-1}}\right)$ and $x_i = \frac{F_i}{F_{i-1}} - 1$ are *log* and *simple* returns over $[t_{i-1}, t_i]$

- Impossible to robustly replicate payoffs $RV_T^{(1)}$ and $RV_T^{(2)}$. But possible for something close enough:

$$RV_T^{(3)} = RV^{(3)}(\mathcal{T}) := \sum_{i=1}^{n} 2\left(e^{r_i} - 1 - r_i\right).$$

- "Modified" realized variance (or, realized entropy) $RV_T^{(3)}$ looks strange, but is strictly positive and very similar to $RV_T^{(1)}$ and $RV_T^{(2)}$

$$RV_T^{(3)} \approx \frac{2}{3} RV_T^{(1)} + \frac{1}{3} RV_T^{(2)}$$

THREE FUNCTIONS

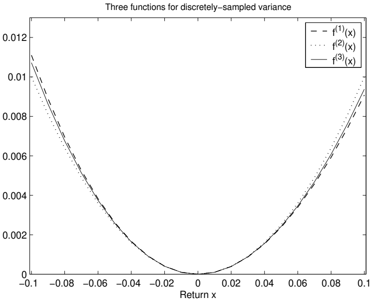


FIGURE: Functions $f^{(1)}(x) = [\ln(1+x)]^2, f^{(2)}(x) = x^2$, and $f^{(3)}(x) = 2\,(x - \ln(1+x))$ used in definitions of $RV_T^{(1)}$, $RV_T^{(2)}$, and $RV_T^{(3)}$.

- Modified realized variance $RV_T^{(3)}$ satisfies AP and its market price

$$E_0^{\mathbb{Q}}[RV_T^{(3)}] = 2\int_0^\infty \frac{M_0(K,T)}{K^2}dK = MFIV = \text{Ideal } VIX^2$$

- Bondarenko (2004) uses $RV_T^{(3)}$ to document negative VRP for S&P 500 and to show hedge funds routinely sell short volatility

- *Corridor* realized variance – accumulates when F_t is between barriers B_1 and B_2
  - Carr and Madan (1998), Andersen and Bondarenko (2007)
  - Up- and Down-Variance – Andersen and Bondarenko (2011)

- Andersen, Bondarenko, and Gonzalez-Perez (2015), a version that satisfies AP:

$$CRV_T^{(3)} = \sum_{i=1}^{n} 2\left(\frac{F_i}{\overline{F}_i}\left(\frac{\overline{F}_i - \overline{F}_{i-1}}{\overline{F}_{i-1}}\right) - \ln\frac{\overline{F}_i}{\overline{F}_{i-1}}\right)$$

where $\overline{F}$ is the corridor truncation operator

$$\overline{F} = \begin{cases} B_1, & F < B_1 \\ F, & B_1 \leq F \leq B_2 \\ B_2, & F > B_2. \end{cases}$$

- Its market price

$$E_0^{\mathbb{Q}}[CRV_T^{(3)}] = 2\int_{B_1}^{B_2} \frac{M_T(K)}{K^2} dK \approx \text{“Real” } VIX^2$$

- Important advantage: deep OTM puts and calls are now not required

- Power-price weighted variance contracts of Bondarenko (2014) – approximate $\int_0^T F_t^a v_t dt$ for different power a:

a	$A(x)$	$B(x) = -A'(x)$	$\frac{1}{2}A''(x)x^2$	$H(x,y) = A(y) - A(x) + B(x)(y-x)$
-1	$\frac{1}{x}$	$\frac{1}{x^2}$	$\frac{1}{x}$	$(y-x)^2 \, \frac{1}{x^2 y}$
0	$-2\ln(x)$	$\frac{2}{x}$	1	$2\left(\frac{y}{x} - 1 - \ln\left(\frac{y}{x}\right)\right)$
1	$2(x\ln(x) - x)$	$-2\ln(x)$	x	$2\left(y\ln\left(\frac{y}{x}\right) - (y-x)\right)$
2	x^2	$-2x$	x^2	$(y-x)^2$
3	$\frac{1}{3}x^3$	$-x^2$	x^3	$\frac{1}{3}(y-x)^2(2x+y)$

- Special cases:
    - $a = 0$ – “Standard” variance $RV_T^{(3)}$
    - $a = 1$ – “Gamma” variance
    - $a = 2$ – “Simple” variance, Carr and Corso (2001), Martin (2017):

$$\sum_{i=1}^{n}(F_i - F_{i-1})^2$$

- Divergence power swaps, Schneider and Trojani (2019), $a = \frac{1}{2}$; Realized skeweness, Orlowski, Schneider, Trojani (2021)

Proposition: CO payoff function $H(x, y) := B(x, y, K)\,|y - K|$ satisfies AP.

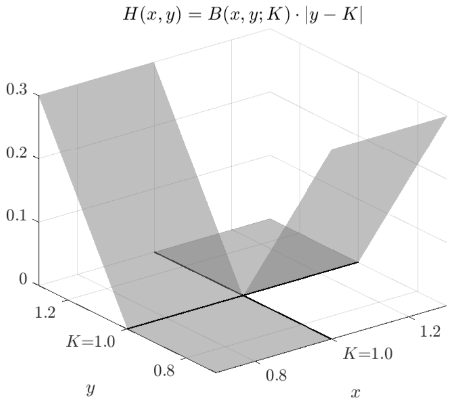


FIGURE: Function $H(x, y)$ in CO payoff when $K = 1.0$.

Proposition: For any $\mathcal{T}$, payoff Φ_T can be perfectly replicated by

(I) a time-T payoff equal to $M_T(K)$;

(II) a dynamic trading strategy, which is rebalanced on dates $t_i \in \mathcal{T}$ to maintain $w_i = U(F_0 - K) - U(F_i - K)$ shares of the underlying.

Therefore,

$$E_0^{\mathbb{Q}}[\Phi_T] = M_0(K).$$

- **Static position** is long one OTM option
- **Dynamic strategy** is binary:
  - If starting **below** the barrier, $F_0 \leq K$

$$w_i = -U(F_i - K) = \begin{cases} 0 & \text{if} \quad F_i \leq K \\ -1 & \text{if} \quad F_i > K \end{cases}$$

  - If starting **above** the barrier, $F_0 > K$

$$w_i = 1 - U(F_i - K) = \begin{cases} 1 & \text{if} \quad F_i \leq K \\ 0 & \text{if} \quad F_i > K \end{cases}$$

- The initial position w_0 is always 0
- Adjusted every time the barrier is crossed over

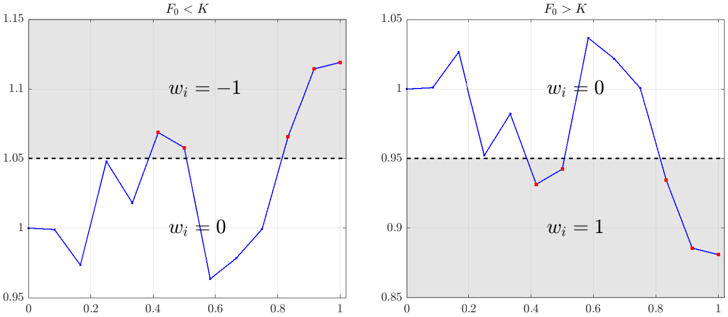


FIGURE: Dynamic replication strategy: $T = 1$ year, $K = 1.05$ or 0.95, and $\Delta t = 1$ month.

Proposition: For any $\mathcal{T}$, payoff $M_T(K,T)$ can be perfectly replicated by

(I) buying n 1-period OTM options with strike K on dates $t_0, t_1, \ldots, t_{n-1}$;

(II) a dynamic trading strategy, which is rebalanced on dates $t_i \in \mathcal{T}$ to maintain $u_i = U(F_i - K) - U(F_0 - K)$ shares of the underlying.

- F.e., a 10-year OTM put (not traded) can be replicated by rolling over ten 1-year OTM options (traded)
- These 1-year options all have same strike K and are OTM on purchase day, but option type (Call or Put) depends on a particular price path

Further Results

Portfolios of CO Options

Proposition: Any payoff that satisfies AP can be viewed as a portfolio of CO options.

- CO options are building blocks to engineer generalized variance contracts

Continuous-Time Limit

Proposition: For a general semimartingale F_t, $M_0(K,T) = E_0^{\mathbb{Q}}\left[\frac{1}{2}\Lambda_T(K) + J_T(K)\right]$

- $\Lambda_t(K)$ is local time process (measures time spent at point K over interval $[0,t]$):

$$\Lambda_t(K) = \lim_{\varepsilon \to 0} \frac{1}{2\varepsilon} \int_0^t 1_{\{K-\varepsilon<F_s<K+\varepsilon\}}\, d[F]_s, \quad \text{where} \quad [F]_t = \int_0^t F_s^2 v_s ds$$

- $J_t(K)$ is a pure jump process. At time s, it increases if
  (I) there is a jump, $\Delta F_s = F_s - F_{s-} \neq 0$
  (2) the jump crosses over the barrier, $B(F_{s-}, F_s, K) = 1$

$$J_t(K) = \sum_{s \le t} \left((F_s - K)^+ - (F_{s-} - K)^+ - 1_{\{F_{s-} > K\}} \Delta F_s \right) = \sum_{s \le t} H(F_{s-}, F_s)$$

- Carr and Jarrow (1990) only consider *continuous* semimartingales. $J_t(K)$ accounts for two types of discontinuities: (i) true jumps, or (ii) non-trading periods

- Many potential applications, both for practitioners and academics
- A CO option pays "variance along barrier K"
- A useful risk-management tool in its own right. Traders can bet on volatility around special price levels (support, resistance)
  - "Pinning risk" – large positions of market makers for a certain strike
  - Useful for mean-reverting assets: VIX, FX, interest rates
- Use CO options as building blocks to engineer generalized variance contracts (say, realized variance swaps)
- Availability of robust replication means market makers will post tight quotes in CO options
- New model-free replication strategy:
  n-year OTM option as a sequence of n 1-year options

Applications

- Can study jumps: Diffusion shocks and jumps have different contributions to CO payoff Φ_T. Two types of discontinuities:
  - Real jumps
  - Non-trading periods
- Since vanilla options *are* CO options, can exploit this connection
- New perspective on "expensive put puzzle"
- Risk-premium for variance along different barriers
- Pricing by Monte-Carlo simulations

- Use Monte-Carlo (MC) to value a vanilla option with no closed-form solution (say, SABR model of Hagan et al (2002))
- Simulate J price histories, compute payoff for each history j, and average them
- **Important:** Can use *any* CO option to construct an unbiased estimator for vanilla option!
- Take $\Delta t = T/n$, Φ_n^j is CO payoff for history j, and

$$\hat{V}_n = \frac{1}{J}\sum_{j=1}^{J}\Phi_n^j$$

  - $n = 1$: traditional MC estimator based on path-independent $\Phi^j = (K - F_T^j)^+$
  - $n > 1$: new estimator based on path-dependent CO payoff

Model	$k=0.95$					$k=1.00$				
	n=1	n=10	n=10^2	n=10^3	AV	n=1	n=10	n=10^2	n=10^3	AV
BS	1.00	2.07	2.43	2.49	3.74	1.00	2.55	3.08	3.15	6.21
JD	1.00	1.68	1.84	1.86	2.25	1.00	2.01	2.26	2.29	3.31
SV	1.00	2.19	2.63	2.70	3.85	1.00	2.71	3.37	3.47	6.12

TABLE: Relative Efficiency of five MC estimators for a put option under BS, JD, and SV models when $T = 1$ year, $k = 0.95$ or 1.00.

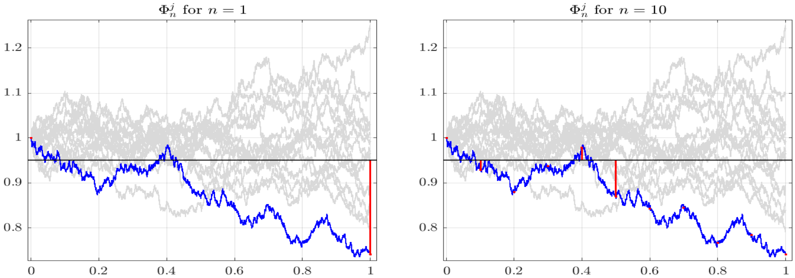
- New estimator is the more accurate, the larger n
- Relative Efficiency (RE) of $\hat{V}_n$ with respect to $\hat{V}_1$ is

$$RE(\hat{V}_n; \hat{V}_1) := \frac{E^{\mathbb{Q}}\left[(\hat{V}_1 - V)^2\right]}{E^{\mathbb{Q}}\left[(\hat{V}_n - V)^2\right]} = \frac{\sigma_1^2}{\sigma_n^2}$$

- The "average" estimator:

$$\hat{V}_{AV} = \frac{1}{4}\hat{V}_1 + \frac{1}{4}\hat{V}_{10} + \frac{1}{4}\hat{V}_{100} + \frac{1}{4}\hat{V}_{1000}$$

- Gain in efficiency is considerable. For ATM put, $\hat{V}_{AV}$ achieves a given accuracy 6.2 times faster than $\hat{V}_1$ under BS model



- Traditional estimator $\hat{V}_1$ uses final price F_T^j only – it is very "wasteful"
- New estimator $\hat{V}_n$ uses n points from each history. Intermediate prices too contain useful information. Estimator is the more efficient, the larger n
- Not obvious. Conditional on F_T, why does it help to know intermediate prices?
- Correlation between Φ_1^j and Φ_n^j is low for high n
- Can do better than $\hat{V}_{AV}$ by using *optimal* weights

Conclusion

- Introduce a new class of path-dependent options – CO options
- A CO payoff satisfies AP and can be robustly replicated
  1) model-free, including jumps
  2) holds exactly for *any* partition $\mathcal{T}$ (non-equidistant, non-small Δt)
  3) consists of two parts:
     (I) a *static* position in vanilla options
     (II) a *discrete* dynamic trading in underlying
- CO options generalize vanilla options, leading to many new results and applications for vanilla options
- It is common to use geographical references to name different types of exotic options: *European*, *American*, *Bermudan*, *Canary*, *Asian*, *Russian*, *Parisian*, *Boston*, etc.

  Maybe refer to CO options as *Ukrainian*?

Thank you!

Access the paper at

https://papers.ssrn.com/abstract_id=4592789